

Conformal Field Theory and the Exact Solution of the BCS Hamiltonian

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We propose a connection between conformal field theory (CFT) and the exact solution and integrability of the reduced BCS model of superconductivity. The relevant CFT is given by the $SU(2)_k$ -WZW model in the singular limit when the level k goes to -2 . This theory has to be perturbed by an operator proportional to the inverse of the BCS coupling constant. Using the free field realization of this perturbed Wess-Zumino-Witten model, we derive the exact Richardson's wave function and the integrals of motion of the reduced BCS model in the saddle point approximation. The construction is reminiscent of the CFT approach to the Fractional Quantum Hall effect.

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I) INTRODUCTION

The BCS theory has been used for decades to describe the superconducting properties of “low T_c ” metallic materials¹. The starting point of the theory is a Hamiltonian which describes the attractive interaction between the electrons in well defined energy levels. The grand canonical BCS wave function gives a very accurate solution of the BCS Hamiltonian in the limit where the number of electrons, N_e , is very large. The BCS wave function may also be projected to a fixed number of electrons, N_e , giving essentially the same physics when $N_e \gg 1$. However for small values of N_e , one has to use exact analytical or numerical methods to obtain reliable results. The study of small fixed- N_e superconductivity has a long story which goes back to an old question posed by Anderson as to what is the smaller size of a metallic particle to remain superconducting². Recent experiments³ involving aluminium grains with nanometer size have inspired a number of theoretical works where Anderson's question is reconsidered^{4–15}.

In this paper we shall be concerned with the exact analytic solution of the reduced BCS Hamiltonian proposed by Richardson in a series of papers between 1963 and 1977^{16–24}. This solution emerged in the framework of Nuclear Physics and has passed unnoticed by most of the physics community until the recent upheaval in ultrasmall metallic grains. A closely related work is that of Cambiaggio, Rivas and Saraceno (CRS) who proved recently the integrability of the reduced BCS Hamiltonian without recourse to Richardson's solution²⁵. These authors found a set of integrals of motion whose number equals that of the degrees of freedom of the system. Our aim is to show that Richardson's solution, together with its integrability, can be naturally understood in the framework of conformal field theory (CFT). In other words, we shall propose a correspondence between CFT and the BCS theory which gives a neat picture of the Richardson's wave function and the conserved quantities found

by CRS, which are in that sense unified in a common framework. Our work is reminiscent to the application of CFT to the Laughlin wave function²⁷ of the Fractional Quantum Hall effect (FQHE)²⁸.

The organization of the paper is as follows. In sections II and III we present brief reviews on the exact solution of the reduced BCS Hamiltonian and the free field realization of the $SU(2)_k$ -WZW model. In section IV we derive the exact Richardson's solution and we show integrability of the BCS model using the free field realization of the $SU(2)_k$ -WZW model. In section V we explain the BCS/CFT connection in a second quantized language. In section VI we explore the analogies and differences between the CFT approaches to BCS and the FQHE. Finally, in section VII we state our conclusions and prospects of future work.

II) REVIEW OF THE EXACT SOLUTION OF THE BCS MODEL

The reduced BCS model is defined by the Hamiltonian^{1,4–15}

$$H_{BCS} = \sum_{j,\sigma=\pm} \varepsilon_{j\sigma} c_{j\sigma}^\dagger c_{j\sigma} - gd \sum_{j,j'} c_{j+}^\dagger c_{j-}^\dagger c_{j'-} c_{j'+} \quad (1)$$

where $c_{j,\pm}$ (resp. $c_{j,\pm}^\dagger$) is an electron destruction (resp. creation) operator in the time-reversed states $|j, \pm\rangle$ with energies ε_j , d is the mean level spacing and g is the BCS dimensionless coupling constant. The sums in (1) run over a set of Ω doubly degenerate energy levels $\varepsilon_j (j = 1, \dots, \Omega)$. We shall assume in this paper that the energy levels are all distinct, i.e. $\varepsilon_i \neq \varepsilon_j$ for $i \neq j$. The Hamiltonian (1) is really a simplified version of the reduced BCS Hamiltonian where all couplings have been set equal to a single one, namely g . This is the model that is commonly used to describe ultrasmall grains, which describes the scattering of pairs of electrons between discrete energy levels that come in time-reversed

states. Hereafter we shall refer to (1) simply as the BCS Hamiltonian.

As mentioned in the Introduction, Richardson had long ago solved this model exactly for an arbitrary set of levels, ε_j , not necessarily all distinct^{16–24}. To simplify matters, we shall assume that there are not singly occupied electronic levels. As can be seen from (1), these levels decouple from the rest of the system; they are said to be blocked, contributing only with their energy ε_j to the total energy \mathcal{E} . The above simplification implies that every energy level j is either empty (i.e. $|\text{vac}\rangle$), or occupied by a pair of electrons (i.e. $c_{j,+}^\dagger c_{j,-}^\dagger |\text{vac}\rangle$). Denote the total number of electrons pairs by N . Then of course $N \leq \Omega$. The most studied case in the literature corresponds to the half-filled situation, where the number of electrons, $N_e = 2N$, is equal to the number of levels Ω ^{4–15}. In the absence of interaction (i.e. $g = 0$), all the pairs occupy the lowest energy levels forming a Fermi sea. The pairing interaction promotes the pairs to higher energies and eventually, for large values of N , all the levels are pair correlated, giving rise to superconductivity¹.

Richardson's solution

In order to describe Richardson's solution one defines the hard-core boson operators

$$b_j = c_{j,-} c_{j,+}, \quad b_j^\dagger = c_{j,+}^\dagger c_{j,-}^\dagger, \quad N_j = b_j^\dagger b_j \quad (2)$$

which satisfy the commutation relations,

$$[b_j, b_{j'}^\dagger] = \delta_{j,j'} (1 - 2N_j) \quad (3)$$

The Hamiltonian (1) can then be written as

$$H_{BCS} = \sum_j 2\varepsilon_j b_j^\dagger b_j - g \sum_{j,j'} b_j^\dagger b_{j'}, \quad (4)$$

where we have set $d = 1$ (i.e. all the energies are measured in units of d). Richardson showed that the eigenstates of this Hamiltonian with N pairs have the (unnormalized) product form^{16–18} (for a direct proof of the results of this subsection see reference²⁹)

$$|N\rangle_R = \prod_{\nu=1}^N B_\nu |\text{vac}\rangle, \quad B_\nu = \sum_{j=1}^{\Omega} \frac{1}{2\varepsilon_j - e_\nu} b_j^\dagger \quad (5)$$

where the parameters e_ν ($\nu = 1, \dots, N$) are, in general, complex solutions of the N coupled algebraic equations

$$\frac{1}{g} + \sum_{\mu=1(\neq\nu)}^N \frac{2}{e_\mu - e_\nu} = \sum_{j=1}^{\Omega} \frac{1}{2\varepsilon_j - e_\nu}, \quad (6)$$

The energy of these states is given by the sum of the auxiliary parameters e_ν , i.e.

$$\mathcal{E}(N) = \sum_{\nu=1}^N e_\nu \quad (7)$$

The ground state of H_{BCS} is given by the solution of eqs.(6) which gives the lowest value of $\mathcal{E}(N)$. The (normalized) states (5) can also be written as²⁰

$$|N\rangle_R = \frac{C}{\sqrt{N!}} \sum_{j_1, \dots, j_N} \psi^R(j_1, \dots, j_N) b_{j_1}^\dagger \dots b_{j_N}^\dagger |\text{vac}\rangle \quad (8)$$

where the sum excludes double occupancy of pair states and the wave function ψ takes the form

$$\psi^R(j_1, \dots, j_N) = \sum_{\mathcal{P}} \prod_{k=1}^N \frac{1}{2\varepsilon_{j_k} - e_{\mathcal{P}k}} \quad (9)$$

The sum in (9) runs over all the permutations, \mathcal{P} , of $1, \dots, N$. The constant C in (8) guarantees the normalization of the state²⁰ (i.e. ${}_R\langle N|N\rangle_R = 1$); its expression will be given in section IV.

A well known fact about the BCS Hamiltonian is that it is equivalent to that of a XY model with long range couplings and a “position dependent” magnetic field proportional to ε_j . To see this let us represent the hard-core boson operators (2) in terms of the Pauli matrices as follows,

$$b_j = \sigma_j^+, \quad b_j^\dagger = \sigma_j^-, \quad N_j = \frac{1}{2}(1 - \sigma_z) \quad (10)$$

in which case the Hamiltonian (4) becomes

$$H_{BCS} = H_{XY} + \sum_j \varepsilon_j + g(\Omega/2 - N) \quad (11)$$

$$H_{XY} = - \sum_j 2\varepsilon_j t_j^0 - \frac{g}{2}(T^+ T^- + T^- T^+)$$

where the matrices

$$T^a = \sum_{j=1}^{\Omega} t_j^a \quad (a = 0, +, -) \quad (12)$$

$$t_j^0 = \frac{1}{2}\sigma_j^z, \quad t_j^+ = \sigma_j^+, \quad t_j^- = \sigma_j^-$$

satisfy the $SU(2)$ algebra,

$$[T^a, T^b] = f_c^{ab} T^b \quad (13)$$

$$f_+^{+0} = f_-^{0-} = -1, \quad f_0^{+-} = 2$$

whose Casimir is given by

$$\mathbf{T} \cdot \mathbf{T} = T^0 T^0 + \frac{1}{2}(T^+ T^- + T^- T^+) \quad (14)$$

Integrability of the BCS Hamiltonian

From the existence of an exact analytic solution of H_{BCS} , one may expect that H_{BCS} should be integrable. Indeed CRS found the integrals of motion^{25,26},

$$R_i = -t_i^0 - g \sum_{j(\neq i)}^{\Omega} \frac{\mathbf{t}_i \cdot \mathbf{t}_j}{\varepsilon_i - \varepsilon_j}, \quad (i = 1, \dots, \Omega) \quad (15)$$

where the denominator does not blow up since we are assuming non degenerate energy levels. Integrability amounts to the eqs.

$$[H_{BCS}, R_i] = [R_i, R_j] = 0, \quad (i, j = 1, \dots, \Omega) \quad (16)$$

Denote the eigenvalue of R_j acting on the state (8) by λ_j , namely

$$R_j |N\rangle_R = \lambda_j |N\rangle_R \quad (17)$$

CRS, seemingly unaware of Richardson's solution, did not give an expression of λ_j in their work. However, they did show that H_{XY} given in eq.(11) can be expressed in terms of the operators R_i as

$$H_{XY} = \sum_j 2\varepsilon_j R_j + g \left(\sum_j R_j \right)^2 - \frac{3}{4} g \Omega \quad (18)$$

Hence, combining (17) and (18) one can find the eigenvalues of H_{XY}

$$\mathcal{E}_{XY} = \sum_j 2\varepsilon_j \lambda_j + g \left(\sum_j \lambda_j \right)^2 - \frac{3}{4} g \Omega \quad (19)$$

and in turn those of \mathcal{E}_{BCS} by recourse to eq.(11). We shall show in section IV that λ_j has the simple expression

$$\lambda_i = -\frac{1}{2} + g \left(\sum_{\nu=1}^N \frac{1}{2\varepsilon_i - e_\nu} - \frac{1}{4} \sum_{j=1(\neq i)}^{\Omega} \frac{1}{\varepsilon_i - \varepsilon_j} \right) \quad (20)$$

One can check this result by deriving the energy (7) from eqs. (11), (19) and (20).

This ends the presentation of the exact solution of the BCS Hamiltonian. The existence of an underlying analytic structure reminiscent to that of a CFT is apparent from eqs. (9), (15) and (20). Indeed, the aforementioned equations contain factors of the form $1/(z - z')$ where z and z' stands for either $2\varepsilon_j$ or e_ν . Terms of this sort arise quite naturally as correlators (i.e. $\langle A(z)B(z') \rangle = 1/(z - z')$) of chiral primary fields $A(z)$ and $B(z)$ in diverse CFT's. The problem is to identify which CFT explains all the features presented so far in a unified manner. We shall argue that the solution of this problem is given by the Wess-Zumino-Witten (WZW) model based on the affine Kac-Moody group $SU(2)_k$ in the limit where the level k goes to -2 . The proof of this result requires standard tools of CFT and, more precisely, the free field or Coulomb Gas (CG) representation of the WZW model.

III) REVIEW OF THE FREE FIELD REPRESENTATION OF THE $SU(2)_K$ -WZW MODEL

The material presented in this section is standard in the CFT literature³⁰. Nevertheless, we have included it

for the benefit of readers that are not experts in CFT. This will allow us to highlight the main tools we shall use in later sections. We shall follow closely reference³¹.

The WZW model is an interacting theory which nevertheless admits a description in terms of free fields³². The correlators and conformal blocks can then be easily calculated as integrals of vacuum expectation values of vertex operators. This gives an integral representation of the conformal blocks which satisfy automatically the Knizhnik-Zamolodchikov (KZ) equations³³. In the case of the $SU(2)_k$ -WZW model the free fields are a $\beta - \gamma$ system with conformal weights 1 and 0, respectively, and a boson field φ which satisfy the following operator product expansion (OPE)^{31,32},

$$\begin{aligned} \beta(z) \gamma(w) &= -\gamma(z) \beta(w) = \frac{1}{z-w} \\ \varphi(z) \varphi(w) &= -\ln(z-w) \end{aligned} \quad (21)$$

The WZW currents $J^a(z)$ ($a = 0, \pm$) can be expressed in terms of these fields as (hereafter, normal order of operators will be implicitly assumed)

$$\begin{aligned} J^+ &= i \beta \\ J^0 &= -\frac{i}{2\alpha_0} \partial \varphi - \beta \gamma \\ J^- &= i [\beta \gamma^2 + \frac{i}{\alpha_0} \gamma \partial \varphi - k \partial \gamma] \end{aligned} \quad (22)$$

which satisfy the OPE's

$$J^a(z) J^b(w) = \frac{k/2}{(z-w)^2} q^{ab} + \frac{1}{z-w} f_c^{ab} J^c(w) + \text{reg. terms} \quad (23)$$

where f_c^{ab} are the $SU(2)$ structure constants defined in eq.(13), and $q^{00} = 1$, $q^{+-} = q^{-+} = 2$. The level of the WZW model, k , is related to the "charge" α_0 by the eq.

$$k + 2 = \frac{1}{2\alpha_0^2} \quad (24)$$

If k is a positive integer, the WZW model is a Rational Conformal Field Theory (RCFT) with $k+1$ primary fields labelled by the total spin, $j = 0, 1/2, \dots, k/2$. In our CFT approach to BCS, we shall need to consider the limit where $k \rightarrow -2$, which corresponds to taking $\alpha_0 \rightarrow \infty$. This is a singular limit which takes us away from the rational WZW models. Actually, the case when k is exactly -2 is mathematically interesting due to its relation to the singular hyperplanes in the representation theory of affine Kac-Moody algebras^{34,35}. For non-positive integer values of k , we can still define the theory by the free field representation given above.

The Sugawara energy-momentum tensor T_{Sug} of the WZW model is given by the sum of the energy momentum tensors of the $\beta - \gamma$ system and the bosonic field φ

$$T_{Sug} = \beta \partial \gamma - \frac{1}{2} (\partial \varphi)^2 + i \alpha_0 \partial^2 \varphi \quad (25)$$

The central extension c of the Virasoro algebra generated by the modes of $T_{Sug}(= \sum_n L_n z^{-n-2})$ is

$$c = 3 - 12\alpha_0^2 = \frac{3k}{k+2} \quad (26)$$

where the $\beta - \gamma$ system contributes with 2 and the field φ contributes with $1 - 12\alpha_0^2$. In (26) we have used the relation (24), which for integer k 's gives the well known value of the Virasoro central charge of the $SU(2)_k$ -WZW model. In the limit $(k+2) \rightarrow 0$ the central extension c diverges. In order to get a meaningful theory one has to scale the Virasoro operators as $\tilde{L}_n = \lim_{k \rightarrow -2} (k+2)L_n$. In that limit the Virasoro algebra becomes

$$[\tilde{L}_n, \tilde{L}_m] = 0 \quad (27)$$

which suggests some sort of integrability. In fact the commutativity of the Virasoro operators \tilde{L}_n has been used to study the representation theory of the $SU(2)_{k=-2}$ Kac-Moody algebra for the proof of the Kac-Kazhdan conjecture³⁴ concerning character formulas (see³⁵ for references).

The primary fields $\Phi_m^j(z)$ of the WZW model are labelled by the total spin $j = 0, 1/2, \dots$ and the third component of the spin $m = j, \dots, -j$. Their free field representation is given by

$$\begin{aligned} \Phi_m^j(z) &= \gamma^{j-m}(z) V_{\alpha_j}(z) \\ V_{\alpha_j}(z) &= e^{i\alpha_j\varphi(z)}, \quad \alpha_j = -2\alpha_0 j \end{aligned} \quad (28)$$

and have a conformal weight Δ_j given entirely by that of the vertex operator V_{α_j} , namely

$$\Delta_j = \frac{1}{2}\alpha_j(\alpha_j - 2\alpha_0) = \frac{j(j+1)}{k+2} \quad (29)$$

In the free field representation of a CFT, every primary field has a conjugate version besides its "direct representant", which is needed for the computation of correlators³⁶. In the case of the WZW model, the conjugate of the primary field with $m = j$ is

$$\tilde{\Phi}_j^j(z) = \beta^{s+2j}(z) V_{2\alpha_0(s+j)}(z), \quad s = -(k+1) \quad (30)$$

The corresponding equation for $m < j$ is much more complicated, and it is a sum of terms where the difference between β fields and γ fields is given by $s + j + m$.

A particular case of (30) is when $j = 0$, which corresponds to the conjugate field of the identity

$$\tilde{I}(z) = \beta^s(z) V_{2\alpha_0 s}(z) \quad (31)$$

A consequence of (31) is that the expectation values of operators should satisfy the following charge neutrality conditions

$$N_\beta - N_\gamma = s, \quad \sum_i \alpha_i = 2\alpha_0 s, \quad s = -(k+1) \quad (32)$$

where N_β and N_γ is the number of β and γ fields in the correlator, and α_i are the charges of the vertex operators made of the field φ . Eq.(32) means that there is a background charge $-2\alpha_0 s$ in the boson sector and a charge

$-s$ in the $\beta - \gamma$ sector, which need to be neutralize for the correlator to be non-vanishing. The latter properties can alternatively be attributed to the out vacuum which have charges $-2\alpha_0$ and $-s$ in the φ and $\beta - \gamma$ sectors, respectively.

The remaining ingredient of the free field representation is provided by the so called screening charge

$$Q = \oint_C du S(u), \quad S(u) = \beta(u) V_{2\alpha_0}(u) \quad (33)$$

whose basic property is that it commutes with the $SU(2)$ current algebra and the Virasoro operators. In eq.(33) and below, du is meant to contain the factor $1/(2\pi i)$ to takes care of the factor $2\pi i$ that comes out in the residue formula. Using the vertex representations (28) and (30) of the primary fields, together with the screening charge (33), one can compute the conformal blocks of the WZW model. Conformal blocks are the chiral building blocks of correlators. The latter are obtained by combining the holomorphic and the anti-holomorphic conformal blocks and imposing monodromy invariance. In the WZW model a conformal block $\psi^{WZW}(z_1, \dots, z_{\Omega+1})$ involving $\Omega + 1$ primary fields $\{(j_k, m_k)\}_{k=1}^{\Omega+1}$, inserted at the positions $\{z_k\}_{k=1}^{\Omega+1}$, can be associated with the $SU(2)$ tensor product decomposition

$$\begin{aligned} j_1 \otimes \dots \otimes j_\Omega &\rightarrow j_{\Omega+1} \\ \sum_{k=1}^{\Omega} m_k &= m_{\Omega+1} \end{aligned} \quad (34)$$

where $(j_{\Omega+1}, m_{\Omega+1})$ appears as an outgoing state. The free field expression of the conformal blocks is given by

$$\begin{aligned} \psi^{WZW}(z_1, \dots, z_{\Omega+1}) &= \langle \Phi_{m_1}^{j_1}(z_1) \dots \Phi_{m_\Omega}^{j_\Omega}(z_\Omega) \tilde{\Phi}_{m_{\Omega+1}}^{j_{\Omega+1}}(z_{\Omega+1}) \\ &\times \oint_{C_1} du_1 S(u_1) \dots \oint_{C_N} du_N S(u_N) \rangle \end{aligned} \quad (35)$$

where $\tilde{\Phi}_{m_{\Omega+1}}^{j_{\Omega+1}}$ is the conjugate of the outgoing state $(j_{\Omega+1}, m_{\Omega+1})$, and the screening charges are integrated along the contours C_1, \dots, C_N . The charge neutrality conditions (32) applied to (35) yield

$$N = \sum_{k=1}^{\Omega} j_k - j_{\Omega+1}, \quad m_{\Omega+1} = \sum_{k=1}^{\Omega} m_k \quad (36)$$

which agree with the Clebsch-Gordan decomposition (34). The case when $N = 0$ corresponds to the maximal allowed value of $j_{\Omega+1} = \sum_{k=1}^{\Omega} j_k$. On the other hand, if Ω is even, the minimal value of $j_{\Omega+1}$ is zero, which requires $N = \sum_{k=1}^{\Omega} j_k$ screening charges. Hence, the different choices of the screening charges and contours give rise to all possible conformal blocks. In this manner the free field representation provides integral solutions of the KZ equations satisfied by the conformal blocks (35). The KZ eqs are³³

$$\left(\kappa \frac{\partial}{\partial z_i} - \sum_{j=1(\neq i)}^{\Omega+1} \frac{\mathbf{t}_i \cdot \mathbf{t}_j}{z_i - z_j} \right) \psi^{WZW}(z_1, \dots, z_{\Omega+1}) \quad (37)$$

where $\kappa = (k+2)/2$ and \mathbf{t}_i are the $SU(2)$ matrices in the j_i representation acting at the i^{th} site.

IV) CFT REPRESENTATION OF THE EXACT SOLUTION OF BCS

Our first aim is to obtain Richardson's wave function (9) using the free field representation of the WZW model.

Richardson's wave function

The starting point is the pseudospin version of the BCS model introduced in section II, according to which an empty energy level ε_i ($i = 1, \dots, \Omega$) has spin $m_i = 1/2$ while an occupied level has spin $m_i = -1/2$. This suggests to rewrite Richardson's wave function as

$$\psi_{m_1, \dots, m_\Omega}^R(z_1, \dots, z_\Omega; e_1, \dots, e_N) = \sum_{\mathcal{P}} \prod_{k=1}^N \frac{1}{z_{l_k} - e_{\mathcal{P}_k}} \quad (38)$$

where $z_i = 2\varepsilon_i$ and l_k is defined by the condition $m_{l_k} = -1/2$. The $SU(2)$ quantum numbers m_i of (38) satisfy

$$\sum_{i=1}^{\Omega} m_i = \frac{\Omega}{2} - N \quad (39)$$

Let us compare now the conformal block (35) and the wave function (38). If we take $j_k = 1/2$ ($k = 1, \dots, \Omega$) in (35) and use (39), we are led to the identifications

$$j_{\Omega+1} = m_{\Omega+1} = \frac{\Omega}{2} - N \quad (40)$$

which requires $N \leq \Omega/2$. Hence from a formal point of view, we can regard Richardson's wave function as a conformal block involving Ω primary fields $\Phi_{m_j}^{1/2}(z_j)$, located at the positions $z_j = 2\varepsilon_j$, and a primary field $\tilde{\Phi}_{m_{\Omega+1}}^{j_{\Omega+1}}(z_{\Omega+1})$ whose position we shall place at ∞ . The last ingredient we need, in order to reproduce Richardson's wave function using CFT tools, is to find the role played by the BCS coupling constant g . We shall see below that g is associated to the operator

$$V_g = \exp\left(-\frac{i\alpha_0}{g} \oint_{C_g} dz z \partial \varphi(z)\right) \quad (41)$$

Our claim is that Richardson's wave function (38) is given, up to a proportionality factor, by the limit

$$\psi_{\mathbf{m}}^R(\mathbf{z}, \mathbf{e}) \propto \lim_{\alpha_0 \rightarrow \infty} \psi_{\mathbf{m}}^{CG}(\mathbf{z}) \quad (42)$$

where the Coulomb Gas wave function ψ^{CG} is given by the following expectation value

$$\begin{aligned} & \psi_{\mathbf{m}}^{CG}(\mathbf{z}) \\ &= \langle V_g \Phi_{m_1}^{\frac{1}{2}}(z_1) \dots \Phi_{m_\Omega}^{\frac{1}{2}}(z_\Omega) \\ & \times \tilde{\Phi}_{j_{\Omega+1}}^{j_{\Omega+1}}(\infty) \oint_{C_1} du_1 S(u_1) \dots \oint_{C_N} du_N S(u_N) \rangle \end{aligned} \quad (43)$$

Except for the presence of the operator V_g , eq.(43) coincides with the conformal block (35). Let us now prove eq.(42). First of all, using the free field representation of the primary fields, one can write (43) as

$$\psi_{\mathbf{m}}^{CG}(\mathbf{z}) = \oint_{C_1} du_1 \dots \oint_{C_N} du_N \psi^\varphi(\mathbf{z}, \mathbf{u}) \psi_{\mathbf{m}}^{\beta\gamma}(\mathbf{z}, \mathbf{u}) \quad (44)$$

where

$$\psi^\varphi = \langle V_g \prod_{i=1}^{\Omega} V_{-\alpha_0}(z_i) \prod_{\nu=1}^N V_{2\alpha_0}(u_\nu) V_{2\alpha_0(s+j_{\Omega+1})}(\infty) \rangle \quad (45)$$

$$\psi_{\mathbf{m}}^{\beta\gamma} = \langle \prod_{i=1}^{\Omega} \gamma^{\frac{1}{2}-m_i}(z_i) \prod_{\nu=1}^N \beta(u_\nu) \beta^{s+j_{\Omega+1}}(\infty) \gamma^{2j_{\Omega+1}}(\infty) \rangle$$

Using eqs.(21) and the Wick theorem, one can show that $\psi^{\beta\gamma}$ is, up to a sign, equal to the Richardson's wave function, namely

$$\psi_{\mathbf{m}}^R(\mathbf{z}, \mathbf{e}) = (-1)^N \psi_{\mathbf{m}}^{\beta\gamma}(\mathbf{z}, \mathbf{e}) \quad (46)$$

In this equation we have chosen the positions of the screening operators u_i equal to the Richardson's parameters e_i . However, in eq.(44) one must integrate over the u 's. The job of the limit $\alpha_0 \rightarrow \infty$ is to set $u_i = e_i$. Let us see how this happens. First of all the contribution of the vertex operators can be computed using the formula

$$\begin{aligned} & \langle V_g V_{\alpha_1}(w_1) \dots V_{\alpha_M}(w_M) \rangle \\ &= \prod_{i < j} (w_i - w_j)^{\alpha_i \alpha_j} \prod_j e^{-\alpha_0 \alpha_j w_j / g} \end{aligned} \quad (47)$$

where the contour C_g encircles all the coordinates w_j . The factor ψ^φ then becomes

$$\begin{aligned} & \psi^\phi(\mathbf{z}, \mathbf{u}) = \prod_j e^{\alpha_0^2 z_j / g} \prod_\nu e^{-2\alpha_0^2 u_\nu / g} \\ & \times \prod_{i < j} (z_i - z_j)^{\alpha_0^2} \prod_{\nu < \mu} (u_\nu - u_\mu)^{4\alpha_0^2} \prod_{i, \nu} (z_i - u_\nu)^{-2\alpha_0^2} \end{aligned} \quad (48)$$

In the limit $\alpha_0 \rightarrow \infty$, the integral (44) can be computed using the saddle point method. Indeed, writing (48) as

$$\psi^\varphi(\mathbf{z}, \mathbf{u}) = e^{-\alpha_0^2 U(\mathbf{z}, \mathbf{u})} \quad (49)$$

where

$$\begin{aligned} U &= -\sum_{i < j}^{\Omega} \ln(z_i - z_j) - 4 \sum_{\nu < \mu}^N \ln(u_\nu - u_\mu) \\ &+ 2 \sum_{i=1}^{\Omega} \sum_{\nu=1}^N \ln(z_i - u_\nu) + \frac{1}{g} \left(-\sum_{i=1}^{\Omega} z_i + 2 \sum_{\nu=1}^N u_\nu \right) \end{aligned} \quad (50)$$

the stationary solutions of U are given by the solutions of Richardson's eq.(6), namely

$$0 = \left(\frac{\partial U}{\partial u_\nu} \right)_{u_\mu = e_\mu} \rightarrow \frac{1}{g} + \sum_{\mu=1(\neq \nu)}^N \frac{2}{e_\mu - e_\nu} = \sum_{j=1}^{\Omega} \frac{1}{z_j - e_\nu}, \quad (51)$$

Under these conditions the saddle point value of the integral (44) is

$$\psi_{\mathbf{m}}^{CG}(\mathbf{z}) \sim \frac{1}{(2\pi^{1/2}\alpha_0)^N (\det \mathbf{A})^{1/2}} e^{-\alpha_0^2 U(\mathbf{z}, \mathbf{e})} \psi_{\mathbf{m}}^{\beta\gamma}(\mathbf{z}, \mathbf{e}) \quad (52)$$

where \mathbf{A} is the $N \times N$ hessian matrix defined as

$$\mathbf{A}_{\mu,\nu} = -\frac{1}{2} \frac{\partial^2 U}{\partial u_\mu \partial u_\nu} \Big|_{\mathbf{u}=\mathbf{e}} \quad (53)$$

$$A_{\nu,\nu} = \sum_i \frac{1}{(z_i - e_\nu)^2} - \sum_{\nu'}' \frac{2}{(e_{\nu'} - e_\nu)^2}, \quad A_{\mu,\nu} = \frac{2}{(e_\nu - e_\mu)^2}$$

Eqs.(46), (51) and (52) constitute the proof of (42). Moreover the factor $(\det \mathbf{A})^{-1/2}$ in (52) turns out to coincide with the normalization constant C appearing in the normalized state (8)²⁰, namely

$$C = 1/(\det \mathbf{A})^{1/2} \quad (54)$$

Let us make some comments about the results presented so far.

- Richardson has observed that eqs.(6) could be derived as the stationary configurations for a set of electrostatic charges in 2D²⁴. The electrostatic potential of these charges is given by $\alpha_0^2 U$ or rather by the sum $\alpha_0^2(U + U^*)$. Our CFT derivation of the exact BCS solution shows that the value of the charges are $-\alpha_0$ for the energy levels, $2\alpha_0$ for the screening ones, and a charge $2\alpha_0(s + j_{\Omega+1})$ placed at infinity. The sum of all these charges neutralizes the background charge $-2\alpha_0 s$. The stationary conditions are set by the limit $\alpha_0 \rightarrow \infty$, which is equivalent to the limit $k \rightarrow -2$.
- The operator V_g , which is needed in order to get the term $1/g$ in the Richardson's equations, breaks conformal invariance in an explicit manner. This is why ψ^{CG} is not, strictly speaking, a conformal block of the WZW model. In the spirit of Perturbed Conformal Field Theory³⁷ we could say that the WZW model has been perturbed by the chiral operator $-\frac{i\alpha_0}{g} \oint dz z \partial \varphi$ which is equal to $-\alpha_0 a_{-1}/g$, where a_{-1} is the $n = -1$ mode of the field $\partial \varphi$ ($i\partial \varphi(z) = \sum_n a_n z^{-n-1}$).
- The perturbative renormalization group (RG) analysis of the BCS model yields that the coupling constant $g > 0$ flows to large values in the infrared regime, leading to the superconducting instability of the Fermi sea³⁸. In our approach the fixed point of this RG flow is described by the $SU(2)_{k=-2}$ -WZW model. The $1/g$ perturbation takes us away from this fixed point, breaking conformal invariance. In that sense, we are dealing with the strong

coupling version of BCS, which is valid for all values of the BCS coupling constant g , since there are not phase transitions as we go from weak to strong coupling.

- So far, we have assumed for simplicity that the pair degeneracy of the energy levels ε_j is unity, which corresponds in the WZW model to primary fields with spin $j_k = 1/2$. The construction can be straightforwardly generalized to levels with higher degeneracy, $d_k = 1, 2, \dots$, in which case the associated primary fields have spin $j_k = d_k/2$. In this case the Richardson equations read^{23,24},

$$\frac{1}{g} + \sum_{\mu=1(\neq \nu)}^N \frac{2}{e_\mu - e_\nu} = \sum_{j=1}^{\Omega} \frac{d_j}{z_j - e_\nu}, \quad (55)$$

and they can be derived using the free field representation explained above in terms of the fields $\Phi_{m_k}^{j_k}(z_k)$ with $j_k = d_k/2$.

The integrability of BCS from CFT

At this stage it is clear that eqs. (15) and (17) must be related to the KZ eq. (37). After all, in the limit $\alpha_0 \rightarrow \infty$ the CG wave function (43) coincides with the WZW wave function. Let us show this in detail. The KZ eq.(37) with $z_{\Omega+1} = \infty$ and the definition (15) can be written as

$$\left(\kappa \frac{\partial}{\partial z_i} + \frac{1}{2g} R_i + \frac{1}{2g} t_i^0 \right) \psi^{WZW} = 0 \quad (56)$$

Let us define a new wave function ψ

$$\psi^{WZW} = e^{H_{XY}/2g\kappa} \psi \quad (57)$$

Using the commutativity of R_i and H_{XY} and the fact that

$$\frac{\partial}{\partial z_i} H_{XY} = -t_i^0 \quad (58)$$

eq.(56) becomes

$$\left(\frac{1}{2g} R_i + \kappa \frac{\partial}{\partial z_i} \right) \psi = 0 \quad (59)$$

This equation is completely equivalent to the KZ eq. Imposing that ψ diagonalizes R_i with eigenvalue λ_i , ψ becomes proportional to the Richardson solution, or rather to ψ^{CG} . Hence eq.(49) yields the following expression for λ_i

$$\lambda_i = \frac{g}{2} \frac{\partial U}{\partial z_i} \quad (60)$$

which in turn leads to the formula (20). In other words, ψ^{WZW} , ψ^{CG} and ψ^R agree up to overall factors in the

limit when $\alpha_0 \rightarrow \infty$. It is interesting to observe that the factor relating the WZW and Richardson wave functions has indeed a conformal structure reflected solely in the presence of logarithms, namely

$$\begin{aligned} \psi^{WZW} &= e^{\alpha_0^2 (\frac{2}{g} E_{XY} - U)} \psi^R, \\ \frac{2E_{XY}}{g} - U &= \sum_{i < j} \ln(z_i - z_j) \\ &+ 4 \sum_{\nu < \mu} \ln(u_\nu - u_\mu) - 2 \sum_{i, \nu} \ln(z_i - u_\nu) \end{aligned} \quad (61)$$

The Coulomb energy U contains in a disguised manner both the Richardson's eqs. and the constants of motion λ_i .

V) THE BCS/CFT CORRESPONDENCE

In the previous section we have shown the closed relationship between the BCS and the CFT wave functions. In this section we want to investigate this relation at the second quantized level. In a Quantum Field Theory a generic N -body wave function $\Psi(x_1, \dots, x_N)$ is usually constructed from the overlap of a N -body state $|\Psi\rangle$ with the eigenstates $|x_1, \dots, x_N\rangle$ created by the action of the field operator $\hat{\psi}(x)$ acting on the Fock vacuum $|0\rangle$, namely

$$\begin{aligned} \Psi(x_1, \dots, x_N) &= \langle \Psi | x_1, \dots, x_N \rangle \\ |x_1, \dots, x_N\rangle &= \hat{\psi}(x_1) \dots \hat{\psi}(x_N) |0\rangle \end{aligned} \quad (62)$$

An interesting example of this formalism is provided by the CFT interpretation of the Laughlin wave function²⁷ of the Fractional Quantum Hall effect (FQHE), first proposed by Fubini²⁸. In that case $\hat{\psi}(x)$ is a vertex operator of a single boson governed by a $c = 1$ CFT. One can also make use of an array of bosons.

In the spirit of the FQHE/CFT correspondence, one can interpret the Coulomb Gas wave function (43) as

$$\psi_{\mathbf{m}}^{CG}(\mathbf{z}) = \langle \psi^{CG} | z_1, m_1, \dots, z_\Omega, m_\Omega \rangle \quad (63)$$

where

$$\begin{aligned} |z_1, m_1, \dots, z_\Omega, m_\Omega\rangle &= \prod_{i=1}^{\Omega} \Phi_{m_i}^{\frac{1}{2}}(z_i) |0\rangle \\ \langle \psi^{CG} | &= \langle \alpha_0(\Omega - 2N) | V_g \prod_{\nu=1}^N \oint_{C_\nu} du_\nu S(u_\nu) \end{aligned} \quad (64)$$

The out vacuum $\langle \alpha_0(\Omega - 2N) |$ has been defined as the action of the operator $\tilde{\Phi}_{m_{\Omega+1}}^{j_{\Omega+1}}$ on the out vacuum of the WZW model, and has a charge $\alpha_0(\Omega - 2N)$ in the φ sector and no charge in the $\beta - \gamma$ sector. The different states $\langle \psi^{CG} |$ correspond to different choices of the integration contours C_ν of the screening operators $S(u_\nu)$, which in CFT yield different conformal blocks. For example, the ground state is obtained by choosing the ν^{th} -contour ($\nu = 1, \dots, N$) to run from z_ν to infinity, where we assume that the energy levels are ordered in increasing order, i.e. $z_1 < z_2 < \dots < z_\Omega$. The excited states correspond to other contour choices. The total number of eigenstates of

H_{BCS} with N pairs and Ω levels, which is given by the combinatorial number $\binom{\Omega}{N}$, coincides with the total number of contour choices.

Eqs.(63) and (64) provide the basic correspondences between the exact solution of BCS and CFT, which can be extended to other instances. We collect them in table 1 and make some comments below.

BCS	CFT
Pair energy level	WZW Primary field
Pair degeneracy (d_k)	Total spin ($j_k = d_k/2$)
Eigenstates of H_{BCS}	Conformal blocks
Richardson's eqs	Saddle point conditions
Integrability (CRS)	KZ eqs.
$g = \infty$ (g finite)	WZW (Perturbed WZW)
Cooper pair operator	Screening operator
Phase stiffness	$\propto \alpha_0^2$
empty, occupied level	spin up, down
$\prod_{k=1}^N b_{jk}^\dagger \text{vac}\rangle$	$\prod_{i=1}^{\Omega} \Phi_{m_i}^{1/2}(z_i) 0\rangle$
$R(N)$	$\langle \psi^{CG} $
ψ^R	$\psi^{\beta\gamma}$
$b_j, b_j^\dagger, \frac{1}{2} - N_j$	$\oint_{z_j} dz J^+(z), J^-(z), J^0(z)$
C	$1/\sqrt{\det \mathbf{A}}$
λ_i	$\frac{g}{2} \frac{\partial U_C}{\partial z_i}$
R_i	$-g \tilde{L}_{-1}^{(i)}$
H_{XY}	$-g \tilde{L}_0$

Table 1. The BCS/CFT correspondence

- In analogy to eqs.(64) one could try to define the bra state $\langle z_1, m_1, \dots, z_\Omega, m_\Omega |$ using the conjugate operators $\tilde{\Phi}_{m_i}^{\frac{1}{2}}(z)$. However, this state would have a large charge which would be difficult to compensate for. Similarly, the ket state $|\psi^{CG}\rangle$ is not easy to construct since the screening operator $S(z)$ does not have a conjugate version. A possibility would be to use the second screening operator, given by $\tilde{S}(z) = \beta^{s-1} \exp(-i\varphi(z)/\alpha_0)$ ³¹. This operator appears in the construction of fractional spin representations which are related upon a certain reduction to the minimal models. In the limit $\alpha_0 \rightarrow \infty$ we see that $\tilde{S}(z)$ converges to the identity. Further work is needed to clarify this issue.
- In section III we defined the Virasoro operators \tilde{L}_n as the singular limit $\lim_{k \rightarrow -2} (k+2)L_n$. From eq. (59) we see that R_i could be identified with the action of $-g \tilde{L}_{-1}^{(i)}$ on ψ^{CG} , where $\tilde{L}_{-1}^{(i)} = \partial/\partial z_i$. Similarly, eq.(18) implies that H_{XY} could be identified, up to constants, with $-g(k+2) \sum_j z_j \partial/\partial z_j$ and thus with $-g \tilde{L}_0$. Upon these identifications, the integrability of the BCS model, given by eqs. (16), becomes equivalent to the commutativity (27) of the Virasoro operators \tilde{L}_0 and $\tilde{L}_{-1}^{(i)}$. We may expect

the existence of another CRS-like integrals of motion associated to the Virasoro operators $\tilde{L}_n (n \leq -2)$.

- The Richardson's state ${}_R\langle N|$ corresponds, in the CFT formulation, to the state $\langle \psi^{CG}|$, which is the product of N screening operators acting on the out vacuum $\langle \alpha_0(\Omega - 2N)|$. This correspondence suggests that in the grand canonical (g.c.) ensemble (where the number of pairs is not fixed) the corresponding state should be given by

$$\langle \psi_{g.c.}^{CG}| = \langle 0| \exp\left(\oint dz S(z)\right) \quad (65)$$

(Note that we have assumed the half filled condition $\Omega = 2N$). This state is similar to the BCS state given by¹

$$\langle BCS| = \langle \text{vac}| \exp\left(\sum_k g_k b_k\right) \quad (66)$$

where g_k is the ratio v_k/u_k of the BCS variational parameters. In this sense, the screening charge $\oint dz S(z)$ is the CFT version of the Cooper pair operator $\sum_k g_k b_k$. In CFT it has been argued³⁶ that the screening operators can be exponentiated into the action and that their number is fixed upon imposing the charge neutrality conditions on correlators. This CFT exponentiation corresponds to working in the grand canonical ensemble in BCS.

- In the previous item we argued that $S(z) = \beta(z)\exp(2\alpha_0 i\varphi(z))$ is the CFT analogue of $g_k b_k$. On the other hand, b_k corresponds to the contour integration of $J^+(z) = i\beta(z)$ around the pair energy $z_k = 2\varepsilon_k$. Hence it is natural to associate the BCS variational parameter g_k with $\exp(2\alpha_0 i\varphi(z))$. This means that $2\alpha_0\varphi(z)$ can be associated to the phase of the superconducting order parameter. Shifting $\varphi(z)$ by a constant leads to an overall phase shift of the BCS order parameter. This correspondence yields an insight about the physical meaning of α_0^2 , which seems to be related to the phase stiffness or the superfluid density n_s . Indeed, if we identify the phase of the superconducting order parameter θ with $\alpha_0\varphi$ then the Lagrangian of φ becomes that of a continuum XY model for θ , with α_0^2 playing the role of the superfluid density. Actually α_0^2 appears in the denominator of the Lagrangian while n_s appears in the numerator. However recall that we are working in the energy space so things are inverted. The identification of θ with $\alpha_0\varphi$ is also consistent with the fact that both variables are defined modulo 2π . The limit $\alpha_0 \rightarrow \infty$ therefore corresponds to the limit of very large phase stiffness which in fact leads to the standard BCS theory, where the phase of the superconducting order parameter is rigid and

plays no role in fixing the critical temperature or other observables³⁹. As was shown by Richardson²⁴ eqs.(6) reduce in the bulk limit $N \rightarrow \infty$ to the BCS gap equation and hence the state (8) becomes the fixed N projection of the mean field g.c. BCS state. Finite values of α_0 should lead to non mean field theories with the phase θ playing a dynamical role. It is rather intriguing that models of this sort have already been proposed by several authors for an explanation of high- T_c superconductivity³⁹.

VI) COMPARISON BETWEEN BCS AND THE FQHE

In sections V and VI we noticed some analogies between the CFT approaches to BCS and the FQHE. Let us consider them in some more detail. A common feature is the Coulomb Gas treatment. In the FQHE the CG is associated to the Laughlin wave function of N_e electrons at filling factor $\nu = 1/m$,²⁷

$$\psi_L(w_1, \dots, w_{N_e}) = \prod_{i < j}^{N_e} (w_i - w_j)^m e^{-\frac{1}{4} \sum_{\ell} |w_{\ell}|^2} \quad (67)$$

The norm of (67) can be seen as a classical probability distribution $e^{-\beta U_L}$ of a two-dimensional one-component plasma at fictitious temperature $\beta = 1/m$ and potential energy U_L where²⁷,

$$U_L = -2m^2 \sum_{i < j} \ln|w_i - w_j| + \frac{m}{2} \sum_{\ell}^{N_e} |w_{\ell}|^2 \quad (68)$$

The particles with charge m repel each other with a logarithmic interaction, and they are attracted to the origin by a uniform neutralizing background charge with density $\rho = 1/(2\pi\ell_B^2)$, where ℓ_B is the magnetic length, which has been set equal to one in (67) and (68). For small values of $m (= 3, 5, \dots)$ the electrons form a liquid with uniform density $\rho_e = 1/(2\pi m\ell_B^2)$ which neutralizes the background charge. However, for large values of m , Quantum Montecarlo studies have shown that the Laughlin liquid becomes a solid (i.e. a Wigner crystal) where the positions of the charges are localized.

The comparison between the wave function ψ^{φ} given in eq. (48) and the Laughlin wave function ψ_L suggests a formal identification of the electron positions w_j with the screening positions u_j rather than with the pair energies levels z_j . The reason is that both the u 's and the w 's are subject to integration, while the z 's are held fixed. Following this analogy, we may establish the relations

$$m = 4\alpha_0^2, \quad \text{or} \quad \nu = \frac{k+2}{2} \quad (69)$$

according to which the freezing of the screening charges in the limit $\alpha_0 \rightarrow \infty$ would parallel the Wigner crystal

structure of the FQHE when m is large. On the contrary, for finite values of α_0 the screening charges, which are essentially Cooper pairs, would delocalize becoming a sort of liquid. The discussion at the end of the previous section suggests that this liquid should arise from the fluctuations of the phase of the superconducting order parameter.

Besides these analogies, the Laughlin and BCS Coulomb Gas models differ in the nature of the background charge. In the Laughlin case the charge is two-dimensional, while in the BCS case the linear terms appearing in U (see eq. (50)) can be attributed to a linear uniform density $\rho_\ell \propto \alpha_0/g$ placed at infinity. The latter density creates a uniform electric field $\partial_x(U + U^*)$ along the $x = (z + z^*)/2$ axis. Another difference is that the BCS theory is not really conformal invariant for finite values of g while the Laughlin state has gapless edge excitations described by CFT⁴⁰.

VII) CONCLUSIONS AND PROSPECTS

In this paper we have established a closed relationship between the exact solution of the BCS Hamiltonian and the Coulomb Gas version of the $SU(2)_k$ -WZW model in the singular limit when $k \rightarrow -2$. The Richardson's wave function comes from the $\beta - \gamma$ chiral correlators, while the Richardson's eqs. and the normalization factor of the state arises from the saddle point evaluation of the chiral boson correlators. The BCS coupling constant g enters the construction as a perturbation of the WZW model and breaks conformal invariance. The integrability of the BCS model is related to that of the WZW model through the KZ equations, which has lead us to an expression of the integrals of motion of the BCS model found by Cambiaggio, Rivas and Saraceno. We have proposed a BCS/CFT correspondence which, in many respects, parallels the CFT interpretation of the Fractional Quantum Hall effect. We have conjectured that the singular limit $\alpha_0^2 \rightarrow \infty$ amounts in physical terms to the limit of very strong phase stiffness, which leads to the mean field BCS theory. Finite α_0 generalizations of the BCS model may correspond to non mean field theories where the phase of the superconducting order parameter should have a dynamical role as in some models of high- T_c superconductivity³⁹.

Besides giving new insights into the exact solution of the BCS model, the CFT approach may also help in solving some problems as the computation of observables with the Richardson's exact solution. The finite temperature BCS model is also an interesting problem which one may try to address with CFT tools.

Finally, the BCS/CFT approach can be straightforwardly generalized to any WZW model based on an affine Kac-Moody algebra G_k , where k is the level and G is a semi-simple Lie group or supergroup. As in the $SU(2)$ case, one can use the free field realization of these

models^{32,41}. For the G_k -WZW model the singular limit is given by $k + h \rightarrow 0$, where h is the dual Coxeter number of G ³⁵. The charge and spin independent pairing Hamiltonians studied by Richardson^{23,24} in the context on Nuclear Physics probably belong to this category of models.

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Note added After completion of this work we have been informed by prof. A. Belavin about some related work by H.M. Babujian⁴², where he applies the Bethe ansatz and the Knizhnik-Zamolodchikov eqs. to the Gaudin magnets⁴³. The Gaudin's model is given essentially by the $g \rightarrow \infty$ limit of the reduced BCS model. In fact the Gaudin's Hamiltonians can be identified with $\lim_{g \rightarrow \infty} R_j/g$, where R_j are the CRS conserved quantities define in eq.(15).

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